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TOMOGRAPHIC ALGORITHMS IN THE THEORY OF ELASTICITY†

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The problem of finding the distribution of the elastic moduli in a heterogeneous elastic from the results of a series of mechanical tests under given external effects and measurements of the reactions on the body surface is considered. Two solution algorithms are described and a justification is given. © 1999 Elsevier Science Ltd. All rights reserved.

1. DESCRIPTION OF THE PROBLEM

Let a linearly elastic body which occupies the region Ω be characterized by the tensor of unknown moduli of elasticity ${}^4\hat{a}$ with components $a_{ijkl} = a_{ijkl}(\mathbf{x})$ in a Cartesian system of coordinates; the left superscript 4 in ${}^4\hat{a}$ denotes that the quantity ${}^4\hat{a}$ is a fourth-rank tensor (second-rank tensors will be denoted by a circumflex without a superscript).

Assigning forces $\mathbf{P}$ (displacements $\mathbf{g}$) on the surface Σ of a body Ω and measuring the resulting displacements $\mathbf{g}$ (forces $\mathbf{P}$) the problem is to find the dependence of the components of the moduli of elasticity tensor ${}^4\hat{a}$ on the coordinates.

This inverse problem of the theory of elasticity belongs to the class of so-called inverse coefficient problems which arise in applications, such as flaw detection in manufacturing (quality control), determining the true stiffness characteristics of structural components and interpreting data of geological prospecting.

In a dynamical formulation of the problem, the external effects are taken to be explosive loads (for large objects) or acoustic loads (surface vibrations). Effective solutions have been developed for so-called acoustic media, where there is only one unknown scalar coefficient [1, 2]. The methods of the inverse scattering problem have been proposed to solve the problem of finding the Lamé coefficients of an isotropic inhomogeneous medium from the results of dynamical tests [12].

Problems of finding one scalar coefficient arise in inverse problems of heat conduction, diffusion and impedance tomography [5]. It turns out that the discrete formulation of the solution algorithm for problems of impedance tomography [3], which involve determining one scalar coefficient in the potential equation in non-uniform isotropic conducting media in a static formulation, can be extended to systems of partial differential equations, and to the equations of the theory of elasticity in particular. In addition to that extension, we give a new algorithm based on the duality concept [4].

2. BASIC EQUATIONS AND CONDITIONS

The state of the medium in region Ω is defined by the well-known equilibrium equations

$$-\frac{\partial}{\partial x_j}[a_{ijkl}(\mathbf{x})\varepsilon_{kl}(\mathbf{u})] = F_i; \quad \varepsilon_{ij}(\mathbf{u}) = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right) \tag{2.1}$$

where $\mathbf{u}$ is the displacement vector, ε_{ij} are the components of the strain tensor and F_i is the density of distributed volume forces. Repeated indices denote summation between 1 and 3 (or 2 in the plane problem).

In the tensor notation introduced above, Eqs (2.1) can be written in the form

$$-\nabla\cdot[{}^4\hat{a}\cdot\cdot\hat{\varepsilon}(\mathbf{u})] = \mathbf{F} \tag{2.2}$$

where ∇ is the Hamilton operator; the dot denotes scalar multiplication (convolution).

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We add to (2.2) conditions which relate the action and reaction on the surface Σ

$$\hat{\sigma}\cdot\boldsymbol{\nu}\big|_\Sigma = {}^4\hat{a}\cdot\cdot\hat{\varepsilon}(\mathbf{u})\cdot\boldsymbol{\nu}\big|_\Sigma = \mathbf{P}(\mathbf{x}),\ \ \mathbf{x}\in\Sigma \tag{2.3}$$

$$\mathbf{u}(\mathbf{x})\big|_\Sigma = \mathbf{g}(\mathbf{x}),\ \ \mathbf{x}\in\Sigma \tag{2.4}$$

where $\hat{\sigma}$ is the stress tensor and $\boldsymbol{\nu}$ is the vector of the unit outward normal to the surface Σ.

The problem reduces to determining the vector field $\mathbf{u}(\mathbf{x})$ and the tensor field ${}^4\hat{a}(\mathbf{x})$ in Ω from Eqs (2.2) and conditions (2.3) and (2.4) using the known functions $\mathbf{P}$, $\mathbf{g}$, $\mathbf{F}$. The problem is non-linear, and can be solved using iterative (step) algorithms.

3. ALGORITHM 1

We assign the initial distribution of the moduli of elasticity

$${}^4\hat{a}(\mathbf{x}) = {}^4\hat{a}^{(0)}(\mathbf{x}),\ \ \mathbf{x}\in\Omega \tag{3.1}$$

and solve the first boundary-value problem of the theory of elasticity—the problem with conditions (2.4). Having constructed Green's function $\hat{G} = \hat{G}^{(0)}(\mathbf{x}, \mathbf{x}_0)$, a second-rank tensor which depends on two variables: $\mathbf{x}$—the coordinates of the point of observation, and $\mathbf{x}_0$—the coordinates of the point of application of the concentrated force, we can write the solution in the form

$$\mathbf{u}^{(0)}(\mathbf{x}) = \int_\Omega \hat{G}^{(0)}(\mathbf{x},\mathbf{x}_0)\cdot\mathbf{F}(\mathbf{x}_0)d\Omega_0 - \int_\Sigma \hat{\sigma}(\hat{G}^{(0)}(\mathbf{x},\mathbf{x}_0)\cdot\mathbf{g}(\mathbf{x}_0))d\Sigma_0 \tag{3.2}$$

In general case the surface reactions computed from solution (3.2) are different from the measured reactions $\mathbf{P}$; corrections for the moduli of elasticity are obtained from the discrepancy.

To compute the first approximation, we put

$${}^4\hat{a}^{(1)} = {}^4\hat{a}^{(0)} + \Delta^4\hat{a}^{(1)},\ \ \mathbf{u}^{(1)} = \mathbf{u}^{(0)} + \Delta\mathbf{u}^{(1)} \tag{3.3}$$

(below we shall omit the superscript 1 for simplicity). We substitute expressions (3.3) into the equations of equilibrium (2.2) and linearize with respect to the increments $\Delta^4\hat{a}$, $\Delta\mathbf{u}$; we finally obtain the equation

$$-\nabla\cdot[{}^4\hat{a}^{(0)}\cdot\cdot\hat{\varepsilon}(\Delta\mathbf{u})] = \nabla\cdot[\Delta^4\hat{a}\cdot\cdot\hat{\varepsilon}(\mathbf{u}^{(0)})] \tag{3.4}$$

which relates the as yet unknown increments $\Delta^4\hat{a}$, $\Delta\mathbf{u}$. The role of the assigned external actions is now played by the right-hand side of Eq. (3.4), and the density $\mathbf{F} = 0$.

Subjecting the corrected displacement field $\mathbf{u} = \mathbf{u}^{(0)} + \Delta\mathbf{u}$ to the boundary condition (2.4) and using the constructed Green's function $\hat{G}^{(0)}$, we obtain the following relation between the increments $\Delta^4\hat{a}$, $\Delta\mathbf{u}$

$$\Delta\mathbf{u}(\mathbf{x}) = \int_\Omega \hat{G}^{(0)}(\mathbf{x},\mathbf{x}_0)\cdot\nabla\cdot[\Delta^4\hat{\mathbf{a}}\cdot\cdot\hat{\varepsilon}(\mathbf{u}^{(0)})]d\Omega_0 \tag{3.5}$$

We now equate the discrepancy of the forces

$$\Delta\mathbf{P} = \hat{\sigma}(\mathbf{u}^{(0)})\cdot\boldsymbol{\nu}\big|_\Sigma - \mathbf{P} \tag{3.6}$$

to the values of $\hat{\sigma}(\Delta\mathbf{u})$, where the field $\Delta\mathbf{u}$ is expressed in terms of $\Delta^4\hat{a}$ from formula (3.5). As a result, we obtain the following basic functional equation from which to find $\Delta^4\hat{a}$

$$\hat{\sigma}\left(\int_\Omega \hat{G}^{(0)}(\mathbf{x},\mathbf{x}_0)\cdot\nabla\cdot[\Delta^4\hat{\mathbf{a}}\cdot\cdot\hat{\varepsilon}(\mathbf{u}^{(0)})]\mathbf{d}\Omega_0\right)\cdot\boldsymbol{\nu}\big|_\Sigma = \Delta\mathbf{P}(\mathbf{x}),\ \ \mathbf{x}\in\Sigma \tag{3.7}$$

4. ALGORITHM 2

As in the previous case, we assign a certain initial distribution of the moduli of elasticity and this time solve the second basic problem of the theory of elasticity, with surface forces $\mathbf{P}$ assigned on the boundary (taking the usual care to ensure uniqueness of the solution in the displacements).

Keeping the same notation for Green's function of the problem as before, we write its solution in the form

$$\mathbf{u}^{(0)}(\mathbf{x}) = \int_{\Omega} \hat{\mathbf{G}}^{(0)}(\mathbf{x},\mathbf{x}_0)\cdot \mathbf{F}(\mathbf{x}_0)\mathbf{d}\Omega_0 + \int_{\Sigma} \hat{\mathbf{G}}^{(0)}(\mathbf{x},\mathbf{x}_0)\cdot \mathbf{P}(\mathbf{x}_0)\mathbf{d}\Sigma_0 \tag{4.1}$$

Using (3.3) for the corrections, we again arrive at Eq. (3.4) with linearized boundary condition of the form

$${}^4\hat{a}^{(0)}\cdot\cdot\hat{\varepsilon}(\Delta\mathbf{u})\cdot\boldsymbol{\nu}|_{\Sigma} = -\Delta^4\hat{\mathbf{a}}\cdot\cdot\hat{\varepsilon}(\mathbf{u}^{(0)})\cdot\boldsymbol{\nu}|_{\Sigma} \tag{4.2}$$

The solution of the resulting problem can be expressed in terms of Green's function constructed in the zero approximation by the formula

$$\Delta\mathbf{u}(\mathbf{x}) = \int_{\Omega} \hat{\mathbf{G}}^{(0)}(\mathbf{x},\mathbf{x}_0)\cdot(\nabla\cdot[\Delta^4\hat{\mathbf{a}}\cdot\cdot\hat{\varepsilon}(\mathbf{u}^{(0)})])\mathbf{d}\Omega_0 - \tag{4.3}$$

$$-\int_{\Sigma} \hat{G}^{(0)}(\mathbf{x},\mathbf{x}_0)\cdot[\Delta^4\hat{\mathbf{a}}\cdot\cdot\hat{\varepsilon}(\mathbf{u}^{(0)})\cdot\boldsymbol{\nu}]_{\Sigma}(\mathbf{x}_0)\mathbf{d}\Sigma_0$$

Having determined the discrepancy of displacements for solution (4.1) and equating it to the value of (4.3) on the surface Σ, we find a basic functional from which we can find the correction of the moduli of elasticity

$$\int_{\Omega} \hat{G}^{(0)}(\mathbf{x},\mathbf{x}_0)\cdot(\nabla\cdot[\Delta^4\hat{\mathbf{a}}\cdot\cdot\hat{\varepsilon}(\mathbf{u}^{(0)})])\mathbf{d}\Omega_0 - \tag{4.4}$$

$$-\int_{\Sigma} \hat{G}^{(0)}(\mathbf{x},\mathbf{x}_0)\cdot[\Delta^4\hat{\mathbf{a}}\cdot\cdot\hat{\varepsilon}(\mathbf{u}^{(0)})\cdot\boldsymbol{\nu}]_{\Sigma}(\mathbf{x}_0)\mathbf{d}\Sigma_0 =$$

$$= \mathbf{g}(\mathbf{x}) - \int_{\Sigma} \hat{\mathbf{G}}^{(0)}(\mathbf{x},\mathbf{x}_0)\cdot\mathbf{P}(\mathbf{x}_0)\mathbf{d}\Sigma_0, \quad \mathbf{x}\in\Sigma$$

Naturally in both algorithms the process is repeated after computing the errors $\Delta^4\hat{a}$ until the required accuracy is attained.

5. AN ALGORITHM FOR SOLVING THE EQUATIONS FOR THE INCREMENTS

It is hardly possible to find an exact solution of Eqs (3.7) and (4.4), while any approximate methods based on representing the solution in the form of series on certain bases will lead to a complication in the usual algorithms in this case, as we shall show.

Let $\{\chi_i\}$ be a basis in the space of functions given on the boundary Σ. Since it is mainly algorithmic difficulties that are of interest here, we need not specify the structure of these spaces or the form of the basis. The spaces of the boundary functions must obviously be spaces of the traces of solutions, and so it is necessary to use Sobolev spaces with a fractional index.

At each step of the algorithms a series of problems is solved and the functions χ_i ($i = 1, 2, \ldots$) are equated either to the solutions themselves or forces expressed in terms of them. As a result, a finite or (in theoretical investigations) an infinite sequence of functional equations of the form (3.7) or (4.4) is obtained which have different solutions $\mathbf{u}_i^{(0)}$ for the same values of $\Delta^4\hat{a}^{(0)}$ and $\Delta^4\hat{a}$.

In the finite-dimensional version, instead of integral equations we obtain a set of systems of linear algebraic equations which have different matrices and right-hand sides, for the same unknowns $\Delta^4\hat{a}$. Generally speaking, of course, the system of equations proves to be overdetermined; it can be solved by the least squares method. Numerical application of the technique has shown that it is at this stage that it becomes evident if the original problem is ill-posed. In such cases the first and second order are regularized with an experimental inspection of the optimum combination of regularization parameters and step length β in the correction formula for the moduli of elasticity.

$${}^4\hat{a}^{(1)} = {}^4\hat{a}^{(0)} + \beta\Delta^4\hat{a}$$

Convergence can be proved by a generalization of the methods used to justify the solution of coefficient problems when the solution in the region Ω is assumed known and only the coefficients need to be determined [6, 7]. The solution will be unique if the orientation of the principal axes of the tensor 512g is known.

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